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Some Results in Chebyshev Rational Approximation

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INTRODUCTION

In this paper we consider different kinds of rational approximations. In Theorem 1 we approximate reciprocals of certain entire functions by reciprocals of exponential polynomials under the uniform norm on $[0, +\infty)$. We show by an example that the bound given in Theorem 1 is best possible. In Theorem 2 we consider the question of approximating reciprocals of certain entire functions by reciprocals of linear combinations of certain entire functions of small growth on $[0, +\infty)$. In Theorems 3–8 we consider approximation on $[0, 1]$. In some of these theorems we connect the error of the approximating function with the rate of growth of the function. These results are the analog of the classical ones of S. N. Bernstein ([1, p. 114]).

DEFINITIONS AND NOTATIONS

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function. As usual, we define the order ρ and lower order β $(0 \leqslant \beta \leqslant \rho \leqslant \infty)$ of f as

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^+ \log^+ M(r)}{\log r} = \begin{matrix}\rho \\ \beta\end{matrix} \qquad (0 \leqslant \beta \leqslant \rho \leqslant \infty).$$

If $0 < \rho < \infty$, then we define the type τ and the lower type ω as

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^+ M(r)}{r^\rho} = \begin{matrix}\tau \\ \omega\end{matrix} \qquad \begin{pmatrix} 0 < \rho < \infty \\ 0 \leqslant \omega \leqslant \tau \leqslant \infty \end{pmatrix},$$

where $M(r) = \max_{|z|=r} |f(z)|$.

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Define

$$\lambda_{0,n} = \lambda_{0,n}\frac{1}{f(x)} \equiv \inf_{P_n \in \Pi_n} \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)}, \tag{1}$$

$$R_{0,n} = R_{0,n}\frac{1}{f(x)} \equiv \inf_{P_n \in \Pi_n} \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,1]}, \tag{2}$$

where Π_n denotes the class of all algebraic polynomials of degree at most n.

Throughout our work $\epsilon > 0$ may be different on different occasions; $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots, c_1, c_2, c_3, \ldots,$ are suitable real constants.

THEOREM 1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, *be an entire function of order* $\rho = 2$, *type* τ *and lower type* ω $(1/25 \leqslant \omega \leqslant \tau < \infty)$ *or order* ρ $(2 < \rho < \infty)$, *type* τ *and lower type* ω $(0 < \omega \leqslant \tau < \infty)$. *Then it is not possible to find exponential polynomials of the form* $\sum_{k=0}^{n} b_k e^{kx}$ $(b_k \geqslant 0)$ *for which*

$$\liminf_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} b_k e^{kx}} \right\|_{L_\infty[0,\infty)}^{\rho\omega/n^2\tau} \leqslant e^{-1}. \tag{3}$$

Proof. Let us assume that there exist $\sum_{k=0}^{n} b_k e^{kx}$, $b_k \geqslant 0$, for which (3) is valid. Then for a sequence of values of n

$$\left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} b_k e^{kx}} \right\|_{L_\infty[0,\infty)} \leqslant \exp\left(\frac{-n^2\tau}{\rho\omega}\right). \tag{4}$$

For every large n we can find an r such that

$$f(r) = \exp(n^2\tau/9\rho\omega). \tag{5}$$

Then, according to (4), we must have

$$\sum_{k=0}^{n} b_k e^{kr} < \exp(n^2\tau/7\rho\omega). \tag{6}$$

First we consider the case $\rho > 2$, $0 < \omega \leqslant \tau < \infty$. That is,

$$0 < \omega = \liminf_{r\to\infty} \frac{\log^+ M(r)}{r^\rho} \leqslant \limsup_{r\to\infty} \frac{\log^+ M(r)}{r^\rho} = \tau < \infty.$$

For each $\epsilon > 0$, we can find an $r_0 = r_0(\epsilon)$ such that for all $r \geqslant r_0(\epsilon)$,

$$\omega(1-\epsilon)\, r^\rho \leqslant \log^+ M(r) \leqslant \tau(1+\epsilon)\, r^\rho. \tag{7}$$

Given any $\delta > 1$, we get from (7), $f(r\delta) \geqslant \{f(r)\}^{\delta^\rho(1-\epsilon)\omega/(1+\epsilon)\tau}$. Choose $\delta^\rho\omega = 4\tau$; then using (5) we get

$$f(r4^{1/\rho}\tau^{1/\rho}\omega^{-1/\rho}) \geqslant \exp(4n^2(1-\epsilon)\tau/9\rho\omega(1+\epsilon)). \tag{8}$$

On the other hand, we have by (6)

$$\begin{aligned}\sum_{k=0}^{n} b_k \exp(kr4^{1/\rho}\tau^{1/\rho}\omega^{-1/\rho}) &= \sum_{k=0}^{n} b_k \exp(kr - kr + kr4^{1/\rho}\tau^{1/\rho}\omega^{-1/\rho})\\ &\leqslant \left(\sum_{k=0}^{n} b_k e^{kr}\right) \exp\{nr[(4\tau\omega^{-1})^{1/\rho} - 1]\}\\ &\leqslant \exp\left\{\frac{n^2\tau}{7\rho\omega} + nr[(4\tau\omega^{-1})^{1/\rho} - 1]\right\}\\ &= \exp\left\{\frac{n^2\tau}{7\rho\omega} + nrc_0\right\}. \end{aligned} \tag{9}$$

From the assumption that f is of positive lower type ω, we get for all large $r \geqslant r_1(\epsilon)$ along with (5),

$$\exp(n^2\tau/9\rho\omega) = f(r) \geqslant \exp(r^\rho\omega(1-\epsilon)). \tag{10}$$

From (10), we obtain

$$r \leqslant (n^2\tau/9\rho\omega^2(1-\epsilon))^{1/\rho}. \tag{11}$$

From (9) and (11) we get

$$\sum_{k=0}^{n} b_k \exp\{kr(4\tau\omega^{-1})^{1/\rho}\} \leqslant \exp\left\{\frac{n^2\tau}{7\rho\omega} + n\left(\frac{n^2\tau}{9\rho\omega(1-\epsilon)}\right)^{1/\rho} c_0\right\}. \tag{12}$$

From (9) and (12), we get at $x = r\delta$,

$$\begin{aligned}\exp\left(\frac{-n^2\tau}{\rho\omega}\right) &< \exp\left(-nrc_0 - \frac{n^2\tau}{7\rho\omega}\right) - \exp\left(\frac{-4n^2(1-\epsilon)\tau}{9\rho(1+\epsilon)\omega}\right),\\ &\leqslant \left(1\Big/\sum_{k=0}^{n} b_k e^{kr\delta}\right) - (1/f(r\delta)). \end{aligned} \tag{13}$$

Clearly (13) contradicts (3), hence the result is proved. Similarly for $\rho = 2$ and $\omega \geqslant 0.04$, we get the result. Q.E.D.

The assumption $\omega > 0.04$ can be relaxed to $\omega > 0$, with a careful selection of $f(r)$ in terms of n.

Remarks. (1) It is interesting to note that the bound in Theorem 4 is essentially best possible. For example, let

$$f(t) = \sum_{k=0}^{\infty} (e^{kt}/e^{k^2}).$$

This is an entire function of order $\rho = 2$, $\tau = \omega = \frac{1}{4}$. For this function by the usual technique (cf. [3]), it is easy to show that

$$\limsup_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} (e^{kx}/e^{k^2})} \right\|_{L_\infty[0,\infty)}^{1/n^2} \leqslant e^{-1/2}.$$

Hence our bound is best possible and we have thereby proved: There exist exponential polynomials $g_n(x) = \sum_{k=0}^{n} a_k e^{kx}$ for which

$$\lim_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} a_k e^{kx}} \right\|_{L_\infty[0,\infty)}^{1/n^2} = e^{-1/2}.$$

(2) There is no analog of Theorem 1 for entire functions of order $\rho = 2$ and type $\tau = 0$. For example, let

$$f(z) = 1 + \sum_{k=1}^{\infty} (e^{zk}/(1^1 2^2 3^3 \cdots k^k)).$$

It is not hard to verify that this is an entire function of order $\rho = 2$ and type $\tau = 0$. For this function, using the methods of [3], it is easy to show that

$$\limsup_{n\to\infty} \left\| \frac{1}{\sum_{k=0}^{n} a_k e^{kx}} - \frac{1}{f(x)} \right\|_{L_\infty[0,\infty)}^{1/n^2 \log n} \leqslant \frac{1}{e}.$$

This clearly contradicts Theorem 1.

(3) The following example suggests that the assumption $\rho = 2$, $\tau > 0$ is not sufficient for the conclusion of Theorem 1. Let

$$f(z) = \sum_{k=0}^{\infty} (e^{zp_k}/e^{p_k^2}), \qquad 0 = p_0 < p_1 < p_2 < \cdots < p_k < \cdots,$$
$$\lim_{k\to\infty} (p_{k+1}/p_k) = \infty.$$

This is an entire function of order $\rho = 2$ and type $\tau > 0$. For this function we can show easily

$$\limsup_{n\to\infty} \left\| \frac{1}{\sum_{k=0}^{n} (e^{xp_k}/e^{p_k^2})} - \frac{1}{f(x)} \right\|_{L_\infty[0,\infty)}^{1/p_n^2} = 0.$$

As usual,

$$0 \leqslant \frac{1}{\sum_{k=0}^{n} (e^{xp_k}/e^{p_k^2})} - \frac{1}{\sum_{k=0}^{\infty} (e^{xp_k}/e^{p_k^2})} \leqslant \sum_{k=n+1}^{\infty} \frac{e^{xp_k}}{e^{p_k^2}}$$

$$\leqslant \frac{e^{xp_{n+1}}}{e^{p_{n+1}^2}} \left(\sum_{i=0}^{\infty} \frac{\exp(x(p_{n+1+i} - p_{n+1}))}{\exp(p_{n+1+i}^2 - p_{n+1}^2)} \right).$$

Now let

$$e^x \leqslant e^{p_{n+1}} \left[1 - \left(\frac{p_n}{p_{n+1}} \right)^2 \log \left(\frac{p_{n+1}}{p_n} \right)^2 \right]^{p_{n+1}}.$$

Then clearly

$$0 \leqslant \frac{1}{\sum_{k=0}^{n} (e^{xp_k}/e^{p_k^2})} - \frac{1}{\sum_{k=0}^{\infty} (e^{xp_k}/e^{p_k^2})} \leqslant C_8 \left(\frac{p_n}{p_{n+1}} \right)^{2p_n^2}. \qquad (A_1)$$

On the other hand, for

$$e^x > e^{p_{n+1}} \left[1 - \left(\frac{p_n}{p_{n+1}} \right)^2 \log \left(\frac{p_{n+1}}{p_n} \right)^2 \right]^{p_{n+1}},$$

$$0 \leqslant \frac{1}{\sum_{k=0}^{n} (e^{xp_k}/e^{p_k^2})} - \frac{1}{f(x)}$$

$$\leqslant \frac{e^{p_n^2}}{e^{xp_n}} \leqslant \frac{\exp(p_n^2 - p_n p_{n+1})}{(1 - 2(p_n/p_{n+1})^2 \log(p_{n+1}/p_n))}. \qquad (A_2)$$

From (A_1) and (A_2) we get the required result.

There exist entire functions of infinite order whose reciprocals can be approximated by reciprocals of exponential polynomials with an error $c^{n \log n}$ $(0 < c < 1)$. For example, let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = 1 + \sum_{k=1}^{\infty} \frac{e^{kz}}{2^{\log 2} 3^{\log 3} \cdots k^{\log k}}.$$

This is an entire function of order $\rho = \infty$. By the usual method, it is not hard to show that

$$\limsup_{n \to \infty} \left\| \frac{1}{\sum_{k=0}^{n} a_k e^{kx}} - \frac{1}{f(x)} \right\|_{L_\infty[0,\infty)}^{1/n \log n} < 1.$$

Now we consider the question of approximating reciprocals of certain entire functions by reciprocals of linear combinations of entire functions of small growth.

THEOREM 2. *Let* $f(z) = \sum_{k=0}^{n} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, *be any entire function of order* ρ $(1 \leqslant \rho < \infty)$, *type* τ *and lower type* ω $(0 < \omega \leqslant \tau < \infty)$. *Let* $\phi(z)$ *be any transcendental entire function with nonnegative coefficients satisfying the assumption that*

$$0 < \lim_{r\to\infty} (\log^+ M_\phi(r)/(\log r)^2) = \theta < 1, \qquad \textit{where} \quad M_\phi(r) = \operatorname*{Max}_{|z|=r} |\phi(z)|.$$

Then for every $g_n(x) = \sum_{k=0}^{n} b_k\{\phi(x)\}^k$, *with* $b_k \geqslant 0$, *we have*

$$\liminf_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} b_k\{\phi(x)\}^k} \right\|_{L_\infty[0,\infty)}^{\rho/N} > e^{-\tau/\omega}, \tag{14}$$

where $N = n(\log n)(\log \log n)$.

Proof. Let us assume (14) is not valid; then for infinitely many n,

$$\left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} b_k\{\phi(x)\}^k} \right\|_{L_\infty[0,\infty)} \leqslant \exp\left(\frac{-N}{\rho\omega}\right). \tag{15}$$

By assumption, $f(z)$ is of order ρ $(1 \leqslant \rho < \infty)$, type τ and lower type ω $(0 < \omega \leqslant \tau < \infty)$, i.e.,

$$0 < \omega = \liminf_{r\to\infty} \frac{\log^+ M(r)}{r^\rho} \leqslant \limsup_{r\to\infty} \frac{\log^+ M(r)}{r^\rho} = \tau < \infty.$$

From this we get, as earlier for any $\alpha > 1$, and for all $r \geqslant r_4(\epsilon)$,

$$f(r\alpha) \geqslant \{f(r)\}^{\alpha^\rho(1-\epsilon)\omega/(1+\epsilon)\tau}. \tag{16}$$

For every large $n \geqslant \hat{n}$, we can find an r such that

$$f(r) = \exp \frac{[n(\log n) \log\log n]\tau}{4\rho\omega} = \exp \frac{N\tau}{4\rho\omega}. \tag{17}$$

At that point

$$g_n(r) = \sum_{k=0}^{n} b_k\{\phi(r)\}^k < \exp(N\tau/3\rho\omega). \tag{18}$$

If (18) is not true, then

$$g_n(r) \geqslant \exp(N\tau/3\rho\omega). \tag{19}$$

It is easy to verify that (17) and (19) contradict (15); hence (18) is valid.

Choose $\omega\alpha^\rho = 4\tau$; then we get from (13) and (14)

$$f(r\alpha) \geqslant \exp(N\tau(1-\epsilon)/(1+\epsilon)\rho\omega). \tag{20}$$

On the other hand,

$$g_n(r\alpha) = \sum_{k=0}^{n} b_k\{\phi(r\alpha)\}^k. \tag{21}$$

By the hypothesis of the above theorem for all $r > r_5(\epsilon)$,

$$\exp[\theta(1-\epsilon)\{\log(r\alpha)\}^2] \leqslant \phi(r\alpha) \leqslant \exp[\theta(1+\epsilon)\{\log(r\alpha)\}^2].$$

From this it is easy to deduce that

$$\phi(r\alpha) \leqslant \{\phi(r)\}^{(1+\epsilon)/(1-\epsilon)} \exp[[\theta(1+\epsilon)][(\log\alpha)^2 + 2\log r\log\alpha]]. \tag{22}$$

From (21) and (22) we get

$$\begin{aligned} g_n(r\alpha) &= \sum_{k=0}^{n} b_k\{\phi(r\alpha)\}^k \\ &\leqslant \sum_{k=0}^{n} b_k[\phi(r)]^{(1+\epsilon)/(1-\epsilon)} \exp[(\log\alpha)^2 + 2(\log\alpha)(\log r)]k \\ &\leqslant \exp[(\log\alpha)^2 + 2(\log\alpha)(\log r)]n \sum_{k=0}^{n} b_k\{\phi(r)\}^{(1+\epsilon)k/(1-\epsilon)}. \end{aligned} \tag{23}$$

We choose ϵ so small that

$$\sum_{k=0}^{n} b_k\{\phi(r)\}^{(1+\epsilon)k/(1-\epsilon)} < \exp(N\tau/3\rho\omega) \qquad \text{(cf. (18))}. \tag{24}$$

By assumption, $f(z)$ is of positive lower type ω; therefore we have for all $r \geqslant r_6(\epsilon)$ along with (17),

$$\exp(r^\rho\omega(1-\epsilon)) \leqslant f(r) = \exp(N\tau/4\rho\omega).$$

From this we get

$$r \leqslant (N\tau/4\rho\omega^2(1-\epsilon))^{1/\rho}. \tag{25}$$

Now by (23), (24), and (25) we get

$$g_n(r\alpha) \leqslant \exp\left\{\frac{N\tau}{3\rho\omega} + (\log\alpha)\left[(\log\alpha) + 2\rho^{-1}\log\left(\frac{N\tau}{4\rho\omega^2(1-\epsilon)}\right)\right]\right\}. \tag{26}$$

From (20) and (26), ϵ being very small, we get

$$\begin{aligned} \exp\left(\frac{-N\tau}{\rho\omega}\right) &< \exp\left(\frac{-N\tau}{3\rho\omega} - L\right) - \exp\left(\frac{-N\tau(1-\epsilon)}{\rho\omega(1+\epsilon)}\right) \\ &\leqslant \frac{1}{\sum_{k=0}^{n} b_k\{\phi(r\alpha)\}^k} - \frac{1}{f(x)}, \end{aligned}$$

where

$$L = (\log \alpha)^2 + 2(\log \alpha)\left[\log\left(\frac{N\tau}{4\rho\omega^2(1-\epsilon)}\right)\right]\rho^{-1}.$$

This clearly contradicts (15); hence the theorem is proved.

Remarks. (I) There exist entire functions of infinite order, whose reciprocals can be approximated by reciprocals of $\sum_{k=0}^{n} \{\phi(x)\}^k (k!)^{-1}$ on $[0, +\infty)$, with an error c^n $(0 < c < 1)$. For example, let

$$f(z) = \sum_{k=0}^{\infty} (\{\phi(z)\}^k/k!),$$

where

$$\phi(z) = 1 + \sum_{i=1}^{\infty} (z^i/(1^1 2^2 3^3 \cdots i^i)).$$

Clearly $f(z)$ is an entire function of infinite order. We can show easily that

$$\limsup_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} (\{\phi(x)\}^k/k!)} \right\|_{L_\infty[0,\infty)}^{1/n} < 1.$$

As usual for $0 \leqslant x \leqslant r$,

$$0 \leqslant \frac{1}{\sum_{k=0}^{n} (\{\phi(x)\}^k/k!)} - \frac{1}{f(x)} \leqslant \sum_{k=n+1}^{\infty} \frac{\{\phi(r)\}^k}{k!}. \tag{B$_1$}$$

For sufficiently large r, it is easy to see that

$$\phi(r) \sim \exp((\log r)^2/2(\log\log r)). \tag{B$_2$}$$

Set

$$\exp((\log r)^2/2 \log\log r) = n/ec, \tag{B$_3$}$$

where $c > 1$ and satisfies $e^n > 2^{ce} c^{nce}$.

A simple manipulation based on (B_1), (B_2), and (B_3) gives us

$$\sum_{k=n+1}^{\infty} \left[\exp\left(\frac{(\log r)^2}{2\log\log r}\right)\right]^k (k!)^{-1} \leqslant \frac{1}{c^n}. \tag{B$_4$}$$

On the other hand, for $x \geqslant r$,

$$0 \leqslant \frac{1}{\sum_{k=0}^{n} (\{\phi(x)\}^k/k!)} - \frac{1}{\sum_{k=0}^{n} (\{\phi(r)\}^k/k!)} \leqslant \frac{1}{f(r) - c^n}. \tag{B$_5$}$$

By assumption,

$$f(z) = \exp[\phi(z)];$$

hence

$$f(r) \sim \exp\left[\exp\left(\frac{(\log r)^2}{2 \log \log r}\right)\right].$$

Therefore we obtain from (B_3) and (B_4)

$$\sum_{k=0}^{n} \frac{\{\phi(r)\}^k}{k!} = f(r) - \sum_{k=n+1}^{\infty} \frac{\{\phi(r)\}^k}{k!} \geqslant f(r) - c^n$$

$$\geqslant \exp\left(\frac{n}{ec}\right) - c^n \geqslant c^n. \tag{B_6}$$

Now the required result follows from (B_4), (B_5), and (B_6).

(II) There exist entire functions of the form

$$f(z) = 1 + \sum_{k=3}^{\infty} \frac{\{\phi(z)\}^k}{3^{(\log 3) \log\log 3} 4^{(\log 4) \log\log 4} \ldots k^{(\log k) \log\log k}},$$

where

$$\phi(z) = 1 + \sum_{j=1}^{\infty} (z^j/e^{j^2}).$$

In this example $f(z)$ fails to satisfy the assumptions of Theorem 2, because $\rho = 1$ and $\tau = 0$. But $\phi(z)$ satisfies the assumption of Theorem 2, since

$$\lim_{r\to\infty} (\log M_\phi(r)/(\log r)^2) = \tfrac{1}{4}.$$

By using the technique of [3], it is easy to show that

$$\limsup_{n\to\infty} \left\| \frac{1}{f(z)} - \frac{1}{g_n(z)} \right\|_{L_\infty[0,\infty)}^{1/n(\log n) \log\log n} \leqslant \frac{1}{4}.$$

Recently much attention has been paid (cf. [3–5]) to approximating reciprocals of certain entire functions by reciprocals of polynomials under the uniform norm on $[0, +\infty)$. However, not much is known about the corresponding question on $[0, 1]$; of course, all the upper bounds that are valid for $[0, +\infty)$ are valid for $[0, 1]$, but we look here for better bounds. We prove here the following.

THEOREM 3. *Let $f(z) = \sum_{k=0}^{n} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be analytic in a disc of radius $q > 1$. Then*

$$\limsup_{n\to\infty} [R_{0,n}]^{1/n} \leqslant 1/q. \tag{27}$$

Proof. Since f is analytic, given any $\epsilon > 0$, such that $q - \epsilon > 1$, we can find an $n_0 = n_0(\epsilon)$, such that for all $n \geqslant n_0(\epsilon)$, we have

$$|a_n| \leqslant (q-\epsilon)^{-n}. \tag{28}$$

From the definition of $R_{0,n}$, we have

$$R_{0,n} \leqslant \sum_{k=n+1}^{\infty} |a_k| a_0^{-2}. \tag{29}$$

From (28) and (29), we get for all large $n \geqslant n_0$,

$$R_{0,n} \leqslant a_0^{-2} \sum_{k=n+1}^{\infty} |a_k| \leqslant a_0^{-2}(q-\epsilon)^{-n}(q-\epsilon-1)^{-1}. \tag{30}$$

Since ϵ is arbitrary, (27) follows from (30).

Remark. If $q = \infty$, then $\lim_{n\to\infty}[R_{0,n}]^{1/n} = 0$.

THEOREM 4. *Let $f(x)$ be a continuous function ($\not\equiv 0$) defined on* [0, 1]. *If there exist polynomials $\{P_n(x)\}_{n=0}^{\infty}$ such that*

$$\lim_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,1]}^{1/n} = 0, \tag{31}$$

then f is the restriction to [0, 1] *of an entire function.*

Proof. The proof of this is very similar to the proof given for [5, Theorem 3] except that here we consider the interval [0, 1], whereas in [5] we considered the positive real axis. Q.E.D.

THEOREM 5. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function of order ρ $(0 < \rho < \infty)$. Then*

$$\limsup_{n\to\infty} \frac{n \log n}{\log[1/R_{0,n}]} = \rho.$$

Proof. As earlier,

$$R_{0,n} \leqslant \sum_{k=n+1}^{\infty} |a_k| a_0^{-2}. \tag{32}$$

Since f is an entire function of order ρ $(0 < \rho < \infty)$, for each $\epsilon > 0$ there is an $n_1 = n_1(\epsilon)$ such that [2, p. 8] for all $n \geqslant n_1(\epsilon)$,

$$|a_n|^{1/n} \leqslant 1/n^{1/\rho+\epsilon}. \tag{33}$$

From (32) and (33), we get

$$[R_{0,n}]^{1/n} \leqslant C_1/n^{1/\rho+\epsilon}, \tag{34}$$

from which it is easy to infer, ϵ being arbitrary,

$$\limsup_{n\to\infty} (n \log n/\log(1/R_{0,n})) \leqslant \rho. \tag{35}$$

From (35), we have for all large $n \geqslant n_2(\epsilon)$,

$$R_{0,n} \leqslant C_7 n^{-n/\rho+\epsilon}.$$

Since $f(x)$ is entire, having nonnegative coefficients, we have for all large $n \geqslant n_4$,

$$0 < f(x) \leqslant f(1) < C_2 < C_1 n^{n/\rho+\epsilon} \leqslant R_{0,n}, \qquad 0 \leqslant x \leqslant 1.$$

Now let us pick $P_n^* \in \Pi_n$ which gives least error in the sense of (2); then

$$R_{0,n} = \left\| \frac{1}{f(x)} - \frac{1}{P_n^*(x)} \right\|_{L_\infty[0,1]}. \tag{36}$$

A simple manipulation based on (36) gives us

$$\frac{-f^2(x)}{(1/R_{0,n}) + f(x)} \leqslant P_n^*(x) - f(x) \leqslant \frac{f^2(x)}{(1/R_{0,n}) - f(x)} \qquad 0 \leqslant x \leqslant 1. \tag{37}$$

From (37), it is easy to obtain that

$$\| P_n^* - f \|_{L_\infty[0,1]} \leqslant \frac{C_4}{(1/R_{0,n}) - C_3}, \qquad 0 \leqslant x \leqslant 1. \tag{38}$$

Let

$$E_n \equiv \inf_{P_n \in \Pi_n} \| f - P_n \|_{L_\infty[0,1]}. \tag{39}$$

Then from (38) and (39) we get

$$E_n \leqslant C_4/((1/R_{0,n}) - C_3). \tag{40}$$

A simple calculation based on (40) gives us

$$E_n \leqslant C_6 R_{0,n}. \tag{41}$$

From (41) we get

$$\limsup_{n\to\infty} \frac{n \log n}{\log(1/E_n)} \leqslant \limsup_{n\to\infty} \frac{n \log n}{\log(1/R_{0,n})}. \tag{42}$$

If $f(z)$ is an entire function of order ρ $(0 < \rho < \infty)$, then for any finite interval, it is known [6, Theorem 1] that

$$\limsup_{n\to\infty} (n \log n/\log(1/E_n)) = \rho. \tag{43}$$

From (35), (42), and (43) we get the required result. Q.E.D.

THEOREM 6. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, *and* $a_k \geqslant 0$ $(k \geqslant 1)$, *be an entire function of order* ρ $(0 < \rho < \infty)$, *type* τ $(0 < \tau < \infty)$. *Then*

$$\limsup_{n\to\infty} (n/\rho e)[R_{0,n}]^{\rho/n} = \tau 4^{-\rho}. \tag{44}$$

Proof. Let $q_n(x; 1) \in \pi_n$ denote the best Chebyshev approximation to f in $[0, 1]$, i.e.,

$$\| f - q_n(x; 1)\|_{L_\infty[0,1]} = \inf_{\sigma_n \in \pi_n} \| f - \sigma_n \|_{L_\infty[0,1]} \equiv E_n .$$

Further, let

$$P_n(x; 1) \equiv q_n(x; 1) + E_n \qquad \text{for every} \quad n \geqslant 0;$$

then it is known [5, p. 181] that

$$\left\| \frac{1}{f(x)} - \frac{1}{P_n(x; 1)} \right\| < 2a_0^{-2} E_n , \qquad x \in [0, 1].$$

From this we get

$$R_{0,n} \leqslant 2E_n a_0^{-2}. \tag{45}$$

Since $f(z)$ is an entire function of order ρ $(0 < \rho < \infty)$ type τ $(0 < \tau < \infty)$,

$$\limsup_{n\to\infty} (n/\rho e)\, E_n^{\rho/n} = \tau 4^{-\rho}\,{}^{1} \qquad \text{(cf. [6, Theorem 3]).} \tag{46}$$

From (45) and (46) we obtain

$$\limsup_{n\to\infty} (n/\rho e)[R_{0,n}]^{\rho/n} \leqslant \tau 4^{-\rho}. \tag{47}$$

On the other hand, we get from (41) and (46),

$$\limsup_{n\to\infty} (n/\rho e)[R_{0,n}]^{\rho/n} \geqslant \tau 4^{-\rho}. \tag{48}$$

We have the required result, (44), from (47) and (48).

[1] The interval considered in [6, 7] is $[-1, 1]$.

Remark. If $f(z)$ is of perfectly regular growth (ρ, τ) (cf. [7, p. 45]), then we can replace lim sup by lim in (44). This follows easily from (41) and (45) of Theorem 6 along with (43) of [7].

THEOREM 7. *Let* $f(z) = \sum_{n=0}^{\infty} a_k z^k$, $a_0 > 0$ *and* $a_k \geqslant 0$ $(k \geqslant 1)$, *be any entire function. Then for all large* n,

$$\log R_{0,n} \sim \log E_n .$$

The proof of this follows from (41) and (45). Q.E.D.

THEOREM 8. *Let* $f(z) = \sum_{n=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, *be an entire function satisfying the assumptions that*

$$1 < \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r)}{\log \log r} = \Lambda + 1 < \infty,$$

$$0 < \limsup_{r\to\infty} \frac{\log^+ M(r)}{(\log r)^{\Lambda+1}} = \tau_l < \infty.$$

Then

$$\limsup_{n\to\infty} \frac{n^{\Lambda+1}}{[\log 1/R_{0,n}]^{\Lambda}} = \frac{\tau_l(\Lambda + 1)^{\Lambda+1}}{\Lambda^{\Lambda}} .$$

The proof of this follows from (41) and (45) by using Lemma 7 and Theorem 7 of [6].

REFERENCES

1. S. N. BERNSTEIN, "Leçons sur les propriétés extrêmales et la Meilleure approximation des fonctions analytiques d'une variable réele," Gauthier–Villars, Paris, 1926.
2. R. P. BOAS, "Entire Functions," Academic Press, New York, 1954.
3. P. ERDÖS AND A. R. REDDY, Rational approximation on the positive real axis, *Proc. London Math. Soc.* **31** (1975), 439–456.
4. P. ERDÖS AND A. R. REDDY, Rational approximation, *Advances in Math.* **21** (1976), 78–109.
5. G. MEINARDUS, A. R. REDDY, G. D. TAYLOR, AND R. S. VARGA, Converse theorems and extensions in Chebyshev rational approximation to certain entire functions in $[0, +\infty)$, *Bull. Amer. Math. Soc.* **77** (1971), 460–461; *Trans. Amer. Math. Soc.* **170** (1972), 171–185.
6. A. R. REDDY, Approximation of an entire function, *J. Approximation Theory* **3** (1970), 128–137.
7. A. R. REDDY, Best polynomial approximation to certain entire functions, Addendum, *J. Approximation Theory* **12** (1974), 199–200.
8. G. VALIRON, "Lectures on the General Theory of Integral Functions," Chelsea, New York, 1949.